# Final Exam 2011 Winter Term 2 Solutions

1. (a) Find the radius of convergence of the series:

$$\sum_{k=0}^{\infty}(-1)^k 2^{k+1} x^k.$$

**Solution:** Using the Ratio Test, we get:

$$L = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}x^{k+1}}{(-1)^k 2^{k+1}x^k}\right| = \lim_{k\to\infty}|-2x| = 2|x|.$$

Note that the series converges for $L < 1$, that is, $2|x| < 1$, which is equivalent to $|x| < \frac{1}{2}$. Thus, the radius of convergence is $\frac{1}{2}$.

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \ldots = \frac{1}{1-r}, \quad |r| < 1.$$

Use this fact to evaluate the series in part (a).

**Solution:** We have that:

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k 2^{k+1} x^k &= 2\sum_{k=0}^{\infty}(-1)^k 2^k x^k = 2\sum_{k=0}^{\infty}(-2x)^k \\ &= 2\left(\frac{1}{1-(-2x)}\right) \qquad (\text{ by the geometric series formula}) \\ &= \frac{2}{1+2x}, \qquad \text{for } |x| < 1/2.\end{aligned}$$

(c) Express the Taylor series of the function:

$$f(x) = \ln(1+2x)$$

about $x = 0$ in summation notation.

**Solution:** First note that:

$$f'(x) = \frac{2}{1+2x} = 2\left(\frac{1}{1-(-2x)}\right) = 2\left(\sum_{k=0}^{\infty}(-2x)^k\right) = \sum_{k=0}^{\infty}(-1)^k 2^{k+1} x^k,$$

for $|x| < 1/2$. Integrating terms by terms, we get:

$$\ln(1+2x) = \sum_{k=0}^{\infty}(-1)^k 2^{k+1} \int x^k\, dx + C = \sum_{k=0}^{\infty}(-1)^k 2^{k+1} \frac{x^{k+1}}{k+1} + C,$$

for $|x| < 1/2$. To solve for $C$, for $x = 0$ (which is in the interval of convergence), we get:

$$ln(1) = \sum_{k=0}^{\infty}(-1)^k 2^{k+1} \frac{0^{k+1}}{k+1} + C \Rightarrow 0 = C.$$

Hence,

$$\ln(1+2x) = \sum_{k=0}^{\infty}(-1)^k 2^{k+1} \frac{x^{k+1}}{k+1},$$

for $|x| < 1/2$.

2. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} e^{-\sqrt{k}}$.

**Solution:** Using Integral Test with $f(x) = \frac{1}{\sqrt{x}} e^{-\sqrt{x}}$, we get:

$$\int_1^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx.$$

Using a direct substitution with $u = -\sqrt{x}$, and $du = -\frac{1}{2\sqrt{x}}\, dx$, we get:

$$\lim_{b\to\infty} \int_{-1}^{-\sqrt{b}} -2e^u\, du = \lim_{b\to\infty} -2e^u \,|_{-1}^{-\sqrt{b}} = \lim_{b\to\infty} -2e^{-\sqrt{b}} + 2e^{-1} = 2e^{-1}.$$

Since $\int_1^{\infty} f(x)\, dx$ converges, by the Integral Test, the series also converges.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$.

**Solution:** Using Limit Comparison Test with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges by the

$p$-series Test, we have that:

$$\lim_{k\to\infty}\left|\frac{a_k}{b_k}\right| = \lim_{k\to\infty}\left|\frac{k^4-2k^3+2}{k^5+k^2+k}\cdot k\right| = \lim_{k\to\infty}\left|\frac{1-2/k+2/k^4}{1+1/k^3+1/k^4}\right| = 1.$$

Thus, either both series converge or both diverge. Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty}\frac{k^4-2k^3+2}{k^5+k^2+k}$.

(c) $\sum_{k=1}^{\infty}\frac{2^k(k!)^2}{(2k)!}$.

**Solution:** We use the Ratio Test,

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{2^{k+1}((k+1)!)^2}{(2(k+1))!}\cdot\frac{(2k)!}{2^k(k!)^2}\right| \\ &= \lim_{k\to\infty}\left|\frac{2^{k+1}}{2^k}\cdot\left(\frac{(k+1)!}{k!}\right)^2\cdot\frac{(2k)!}{(2k+2)!}\right| \\ &= \lim_{k\to\infty}\left|\frac{2(k+1)^2}{(2k+1)(2k+2)}\right| = \frac{1}{2} < 1.\end{aligned}$$

Thus, by the Ratio Test, the series converges.

3. (a) Compute the following indefinite integral:

$$\int \sin(\ln(x))\,dx.$$

**Solution:** Using direct substitution with $t=\ln(x)$, (so $x=e^t$), and $dt=\frac{dx}{x}$ (so $dx = e^t\,dt$), we get:

$$\int \sin(\ln(x))\,dx = \int e^t\sin(t)\,dt.$$

Using integration by parts with $u=e^t$, $du=e^t\,dt$, $dv=\sin(t)\,dt$, $v=-\cos(t)$, we get:

$$\int e^t\sin(t)\,dt = -e^t\cos(t) + \int e^t\cos(t)\,dt.$$

For the remaining integral, we use integration by parts again with $u_1 = e^t$, $du_1 = e^t\,dt$, $dv_1 = \cos(t)\,dt$, $v = \sin(t)$, we get:

$$\int e^t \cos(t)\,dt = e^t \sin(t) - \int e^t \sin(t)\,dt.$$

Thus,

$$\int e^t \sin(t)\,dt = -e^t \cos(t) + \int e^t \cos(t)\,dt = -e^t \cos(t) + e^t \sin(t) - \int e^t \sin(t)\,dt$$
$$\Rightarrow 2\int e^t \sin(t)\,dt = -e^t \cos(t) + e^t \sin(t) + C$$
$$\Rightarrow \int e^t \sin(t)\,dt = -\frac{e^t \cos(t)}{2} + \frac{e^t \sin(t)}{2} + C.$$

In terms of $x$, we get:

$$\int \sin(\ln(x))\,dx = -\frac{x\cos(\ln(x))}{2} + \frac{x\sin(\ln(x))}{2} + C.$$

(b) Evaluate the following definite integral:

$$\int_0^1 \frac{1}{x^2 - 5x + 6}\,dx.$$

**Solution:** Note that the denominator $x^2 - 5x + 6$ can be factored as $(x-2)(x-3)$, which are zero at $x = 2$ and $x = 3$. So, the integrand is, in fact, continuous on the interval $[0, 1]$, and this is not an improper integral. Using partial fractions, we get:

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3) + B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}.$$

So, $A + B = 0$ and $-3A - 2B = 1$. We get $A = -B$, and thus, $-3A - 2B = -A = 1$ yields $A = -1$ and $B = 1$. Hence,

$$\int_0^1 \frac{1}{x^2 - 5x + 6}\,dx = \int_0^1 \left(-\frac{1}{x-2} + \frac{1}{x-3}\right) dx = (-\ln|x-2| + \ln|x-3|)\,|_0^1$$
$$= (-\ln 1 + \ln 2) - (-\ln 2 + \ln 3) = 2\ln 2 - \ln 3.$$

4. Consider the function:

$$F(x) = \begin{cases} a, \text{ if } x < 0, \\ k \arctan x, \text{ if } 0 \leq x \leq 1, \\ b, \text{ if } x > 1. \end{cases}$$

(a) Find the values of $a, k$ and $b$ for which $F$ is a valid cumulative distribution function of a continuous random variable. Then sketch the graph of $F$.

**Solution:** Firstly, for a cumulative distribution function, we need:

- $0 = \lim_{x \to -\infty} F(x) = a$, so $a = 0$.
- $1 = \lim_{x \to \infty} F(x) = b$, so $b = 1$.
- $F(x)$ is continuous on $\mathbb{R}$. In particular, in order to be continuous at $x = 1$, then:

$$\begin{aligned} \lim_{x \to 1^-} F(x) &= F(1) = \lim_{x \to 1^+} F(x) \\ k \arctan(1) &= 1 \\ \Rightarrow k &= \frac{4}{\pi}. \end{aligned}$$

- $F(x)$ is a non-decreasing function since its derivative

$$F'(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{4}{\pi(1+x^2)}, \text{ if } 0 \leq x \leq 1, \\ 0, \text{ if } x > 1. \end{cases}$$

  is non-negative for all $x$.

So, $a = 0$, $b = 1$, and $k = 4/\pi$.

(b) Let $X$ be a continuous random variable with cumulative distribution function $F(x)$ as given in part (a). Find the probability density function of $X$.

**Solution:** Note that if $f(x)$ is the probability density function of $X$, then $f(x) = F'(x)$, so:

$$f(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{4}{\pi(1+x^2)}, \text{ if } 0 \leq x \leq 1, \\ 0, \text{ if } x > 1. \end{cases}$$

5. (a) If

$$F(x) = \int_0^x \ln(2+\sin(t))\,dt \text{ and } G(y) = \int_y^0 \ln(2+\sin(t))\,dt,$$

find $F'\left(\frac{\pi}{2}\right)$ and $G'\left(\frac{\pi}{2}\right)$.

**Solution:** Using the Fundamental Theorem of Calculus Part I, we have:

$$F'(x) = \ln(2+\sin(x)) \Rightarrow F'\left(\frac{\pi}{2}\right) = \ln(2+\sin\left(\frac{\pi}{2}\right)) = \ln(3),$$
$$G'(y) = -\ln(2+\sin(y)) \Rightarrow G'\left(\frac{\pi}{2}\right) = -\ln(2+\sin\left(\frac{\pi}{2}\right)) = -\ln(3).$$

(b) Now define:

$$H(x,y) = \int_y^x \ln(2+\sin t)\,dt.$$

Find the first derivatives of $H$ and use this to compute $\nabla H\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

**Solution:** Using the Fundamental Theorem of Calculus Part I, we get:

$$H_x(x,y) = \ln(2+\sin(x)), \qquad H_y(x,y) = -\ln(2+\sin(y)).$$

We have that:

$$\begin{aligned}\nabla H\left(\frac{\pi}{2},\frac{\pi}{2}\right) &= \langle H_x\left(\frac{\pi}{2},\frac{\pi}{2}\right), H_y\left(\frac{\pi}{2},\frac{\pi}{2}\right)\rangle \\ &= \langle \ln(2+\sin\left(\frac{\pi}{2}\right)), -\ln(2+\sin\left(\frac{\pi}{2}\right))\rangle \\ &= \langle \ln(3), -\ln(3)\rangle.\end{aligned}$$

6. According to market research, the demand curve for a local pizza restaurant satisfies the following relation: if $p$ is the price of a pizza (in dollars), and $q$ is the number of pizzas sold per day, then

$$p^2 + 4q^2 = 800.$$

The restaurant owners want to determine what price the restaurant should charge for each pizza in order to make their daily revenue as high as possible.

(a) Formulate this as a constrained optimization problem, clearly stating the objective function and the constraint.

**Solution:** We want to find the value of $p$ such that the objective function $R(p,q) = pq$ is maximized subject to the constraint that $g(p,q) = p^2 + 4q^2 - 800 = 0$.

(b) Use the method of Lagrange multipliers to solve the problem in part (a). There is no need to justify that the solution you obtained is the absolute maximum or minimum. **A solution that does not use the method of Lagrange multipliers will receive no credit, even if the answer is correct.**

**Solution:** Using the method of Lagrange multipliers, we need to solve the following system:

$$\begin{cases} R_p = \lambda g_p, \\ R_q = \lambda g_q, \\ g(p,q) = 0, \end{cases} \Rightarrow \begin{cases} q = \lambda 2p, \\ p = \lambda 8q, \\ p^2 + 4q^2 - 800 = 0, \end{cases}$$

where $p > 0$ and $q > 0$. Substituting $p = \lambda 8q$ into $q = \lambda 2p$, we get:

$$q = \lambda 2(\lambda 8q) = q\lambda^2 16 \Rightarrow q(1 - 4\lambda)(1 + 4\lambda) = 0.$$

Note that $q = 0$ is not in the domain, so we only consider two cases:

- If $\lambda = 1/4$, then $p = \lambda 8q = 2q$. So, $p^2 + 4q^2 - 800 = 8q^2 - 800 = 0$ which yields $q = 10$, and $p = 20$.
- If $\lambda = -1/4$, then $p = \lambda 8q = -2q$. So, $p^2 + 4q^2 - 800 = 8q^2 - 800 = 0$, and $q = 10$, and $p = -20$ (which is not in the domain since $p$ must be positive).

Thus, the only solution we get is $q = 10, p = 20$ and $\lambda = 1/4$. So, the restaurant should charge \$20 for each pizza in order to maximize daily revenue subject to the constraint that $p^2 + 4q^2 = 800$.

7. (a) Find all critical points of the function:

$$f(x,y) = xye^y + \frac{1}{2}x^2 = 2.$$

**Solution:** Note that $f(x,y)$ is defined for all pairs $(x,y)$ in $\mathbb{R}^2$, so the critical points of $f(x,y)$ are those such that $f_x = f_y = 0$. First, we compute the first order partial derivatives:

$$f_x(x,y) = ye^y + x, \qquad f_y(x,y) = xe^y + xye^y = (y+1)xe^y.$$

Observe that $f_y(x,y) = 0$ for $y = -1$ or $x = 0$. If $x = 0$, then $f_x(0,y) = ye^y = 0$ for $y = 0$. If $y = -1$, then $f_x(x,-1) = -e^{-1} + x = 0$ for $x = e^{-1}$. Thus, we get two critical points $(0,0)$ and $(e^{-1},-1)$.

(b) Classify each critical point you found as a local maximum, local minimum, or a saddle point of $f(x,y)$.

**Solution:** To classify the critical points, we need to find the second order partial derivatives:

$$\begin{aligned} f_{xx}(x,y) &= 1, \\ f_{yy}(x,y) &= xe^y + (y+1)xe^y = (y+2)xe^y, \\ f_{xy}(x,y) &= (y+1)e^y. \end{aligned}$$

Recall that the discriminant is $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$. Using the Second Derivative Test, we get:

- At $(0,0)$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 1$, and $D(0,0) = -1 < 0$. So, $(0,0)$ is a saddle point.
- At $(e^{-1},-1)$, $f_{yy}(e^{-1},-1) = e^{-2}$, $f_{xy}(e^{-1},-1) = 0$, and $D(e^{-1},-1) = e^{-2} > 0$. Since $f_{xx}(e^{-1},-1) = 1 > 0$, the point $(e^{-1},-1)$ is a local minimum.

8. (a) The Maclaurin series for $\arctan(x)$ is given by:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

**Solution:** Note that:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}}\\ &= \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}\\ &= \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)\\ &= \sqrt{3}\left(\frac{\pi}{6}\right).\end{aligned}$$

(b) Find the limit, if it exists, of the sequence $\{a_k\}$, where

$$a_k = \frac{k!\sin^3 k}{(k+1)!}.$$

**Solution:** Since $-1 \leq \sin^3 k \leq 1$, we have:

$$\begin{aligned}-\frac{1}{k+1} &\leq \frac{\sin^3 k}{k+1} \leq \frac{1}{k+1}\\ -\frac{1}{k+1} &\leq \frac{k!\sin^3 k}{(k+1)!} \leq \frac{1}{k+1}.\end{aligned}$$

Since $\lim_{k\to\infty} -\frac{1}{k+1} = \lim_{k\to\infty}\frac{1}{k+1} = 0$, by Squeeze Theorem, we have that:

$$\lim_{k\to\infty}\frac{k!\sin^3 k}{(k+1)!} = 0.$$

(c) Solve the differential equation:

$$\frac{dy}{dx} = xe^{x^2-\ln(y^2)}.$$

**Solution:** We have:

$$\frac{dy}{dx} = \frac{xe^{x^2}}{y^2} \Rightarrow y^2\,dy = xe^{x^2}\,dx.$$

Integrating each side separately, we get:

$$\int y^2\,dy = \frac{y^3}{3} + C.$$

Using a direct substitution $u = x^2$ and $du = 2x\,dx$, we get:

$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^{x^2} + C.$$

Thus, a general solution to the differential equation is:

$$\frac{y^3}{3} = \frac{1}{2}e^{x^2} + C.$$

(d) Identify and sketch the level curve corresponding to $z = e$ of the function:

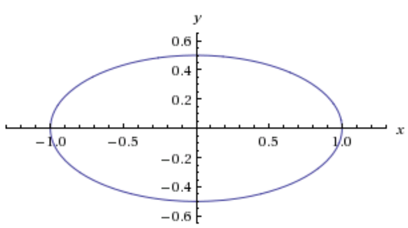
$$z = e^{x^2+4y^2}.$$

Label the axes of your graph and plot the coordinates of at least four points on the level curve.

**Solution:** The level curve corresponding to $z = e$ is:

$$e = e^{x^2+4y^2} \Rightarrow 1 = x^2 + 4y^2,$$

which is an ellipse centered at $(0, 0)$ with $x$-intercepts $(\pm 1, 0)$, and $y$-intercepts $(1, \pm 1/2)$.




(e) There are two boxes containing two balls each. The balls in the first box are numbered $-1$ and 1, the balls in the second box are numbered 0 and 2. An experimenter draws a ball from each box and observes the number of each ball. Define a random variable X whose value is two times the number of the ball drawn from the first box plus three times the number of the ball drawn from the second box. In other words,

$$\text{X} = 2(\text{number observed from box 1}) + 3(\text{number observed from box 2}).$$

Write down all possible values of $X$ and use this to compute the expected value of $X$.

**Solution:** The possible outcomes are: $X = -2$ (with probability $1/4$), $X = 4$ (with probability $1/4$), $X = 2$ (with probability $1/4$), and $X = 8$ (with probability $1/4$). So, the expected value of $X$ is:

$$\mathbb{E}(X) = 1/4(-2) + 1/4(4) + 1/4(2) + 1/4(8) = 3.$$

(f) Find a bound for the error in approximating:

$$\int_0^1 [e^{-2x} + 3x^3]\, dx$$

using Simpson's Rule with $n = 6$ subintervals. There is no need to simplify the answer. **Do not write down the Simpson's rule approximation $S_n$.**

**Solution:** So, $a = 0$, $b = 1$, $n = 6$ and $f(x) = e^{-2x} + 3x^3$. Thus, $\Delta x = \frac{b-a}{n} = \frac{1}{6}$. We have:

$$f'(x) = -2e^{-2x} + 9x^2, \qquad f''(x) = 4e^{-2x} + 18x,$$
$$f'''(x) = -8e^{-2x} + 18, \qquad f^{(4)}(x) = 16e^{-2x}.$$

So, $|f^{(4)}(x)| = 16e^{-2x}$. For $0 \leq x \leq 1$, then $1 \leq e^{2x} \leq e^2$, and $1 \geq e^{-2x} \geq e^{-2}$. Thus, $16 \geq 16e^{-2x} \geq 16e^{-2}$, and we may choose $K = 16$. Hence,

$$E_4 \leq \frac{K(b-a)(\Delta x)^n}{180} = \frac{16}{180(6^6)}.$$

(g) For a certain function $f(x)$, the following equation holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1 - \frac{k^2}{n^2}} = \int_0^1 f(x)\, dx.$$

Find $f(x)$.

**Solution:** First, we recognize that the expression on the left hand side is that of a right Riemann sum, which should have the form:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

So, by comparing with the given expression on the left hand side, we can conclude that $f(x_k^*)\Delta x = \frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}}$. Furthermore, we have $a=0$, $b=1$, and $\Delta x = \frac{b-a}{n} = \frac{1}{n}$. So, $f(x_k^*) = \frac{k}{n}\sqrt{1-\frac{k^2}{n^2}}$. For the right Riemann sum, $x_k^* = x_k = a + k\Delta x = \frac{k}{n}$. So,

$$f(x_k^*) = \frac{k}{n}\sqrt{1-\frac{k^2}{n^2}} = x_k^*\sqrt{1-(x_k^*)^2}.$$

Thus, the function $f(x)$ itself is: $f(x) = x\sqrt{1-x^2}$.

(h) Evaluate the indefinite integral:

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx.$$

**Solution:** We use trigonometric substitution $x = \frac{2}{5}\sec(\theta)$, and $dx = \frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta$. Then,

$$\sqrt{25x^2-4} = 5\sqrt{x^2-4/25} = 5\sqrt{4/25\sec^2(\theta)-4/25} = 2\tan(\theta),$$

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}\,dx &= \int \frac{2\tan(\theta)}{2/5\sec(\theta)}\frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta \\ &= \int 2\tan^2(\theta)\,d\theta = \int (2\sec^2(\theta)-2)\,d\theta \\ &= 2\tan(\theta) - 2\theta + C.\end{aligned}$$

Since $x = \frac{2}{5}\sec(\theta)$, we get $\cos(\theta) = \frac{2}{5x}$, and $\theta = \arccos\left(\frac{2}{5x}\right)$. In a right-angled triangle, the adjacent side to $\theta$ is 2, and the hypotenuse is $5x$, so the opposide side is $\sqrt{25x^2-4}$. Thus, $\tan(\theta) = \frac{\sqrt{25x^2-4}}{2}$. Therefore,

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \sqrt{25x^2-4} - 2\arccos\left(\frac{2}{5x}\right) + C.$$

(i) Find an equation of the plane parallel to $3x - y + 4z = 13$ passing through the point $(2, 1, -1)$.

**Solution:** The normal vector of the plane given by $3x-y+4z=13$ is $\langle 3,-1,4\rangle$. For planes that are parallel to $3x-y+4z=13$, we may use the same normal vector to describe them. Thus, an equation of the plane parallel to $3x-y+4z=13$ passing through the point $(2,1,-1)$ is:

$$3(x-2)-(y-1)+4(z+1)=0 \Leftrightarrow 3x-y+4z=1.$$